APPROXIMATION AND PARAMETER IDENTIFICATION FOR DAMPED SECOND ORDER SYSTEMS WITH UNBOUNDED INPUT OPERATORS

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Abstract

We consider a class of parameter estimation problems motivated by smart structures i.e. structures with integrated piezoelectric actuators and sensors. Problems involving damped second order partial differential equations with unbounded input coefficients are discussed in the context of a variational formulation. Theoretical, computational and experimental aspects of a smart structural system are successfully considered here, illustrating the importance of the proposed procedure in providing accurate models for identification and control. Approximation techniques are introduced and convergence arguments are presented rigorously. Numerical results of parameter estimation procedures are given and experimental data are used to test our computational results.

I. Introduction

High fidelity dynamic models for use in identification and control algorithms are important to current efforts in understanding and design of smart material structures. Particular cases of interest to us here are structures with embedded piezoelectric actuators and sensors. In addition to accurate models, which in most applications are inherently distributed in nature, computational methods (based on PDE approximation ideas) are needed. Parameter estimation techniques are of fundamental interest in model development efforts for the use of piezoelectric materials in such diverse areas as acoustic noise suppression and nondestructive evaluation of materials as well as the more traditional applications involving structural vibration suppression.

For the class of problems we consider here, a cantilevered beam with piezoelectric ceramic patches for actuation and sensing, current models for piezoelectric materials lead to a system with unbounded (in usual state space formulations) input coefficients. These input coefficients, which are related to excitation of moment producing patches, involve derivatives of the delta function.

Our choice of structure is motivated by its simplicity and its representative nature. This configuration has also been well studied by conventional approaches such as finite element methods and provides a standard test bed model for comparison. Several previously published results [?, ?, ?] have indicated that accurate models do not result from first principle without using experimentally updated models. Here we propose a procedure which starts from first principles and retains a rigorous development which does not require updating (or "fudging") to produce an accurate model. This is a significant improvement over previous methodology and is shown here to stand up to repeated experiments with different hardware configurations for sensing and actuation. Previous literature has only claimed success for one configuration of sensors and actuators. Confidence in the model across a variety of configurations is required for design and control. In particular, the methodology used here underscores the importance of the local changes in modulus and damping mechanisms at the location of the piezoelectric patch. Such effects have not previously been considered, and are shown here to provide extremely consistent models. The structure and model reveal the difficulties and possibilities inherent in developing models and methods for more complex structures containing piezoelectric materials.

We consider a cantilevered Euler-Bernoulli beam of length ℓ fixed at $x = 0$ and free at $x = \ell$. The transverse vibrations $y = y(t, x)$ are described by the system

$$\begin{aligned} &\rho(x)\frac{\partial^2 y}{\partial t^2}(t,x) + \gamma\frac{\partial y}{\partial t}(t,x) + \frac{\partial^2 M}{\partial x^2}(t,x) = 0 \qquad 0 < x < \ell,\ t > 0, \\ &y(t,0) = \frac{\partial y}{\partial x}(t,0) = 0, \quad M(t,\ell) = \frac{\partial M}{\partial x}(t,\ell) = 0, \end{aligned} \tag{1}$$

equ:1.1

where $\rho(x)$ is the linear mass density, γ is the coefficient of viscous (air) damping and $M(t, x)$ is the internal moment. For a simple Euler-Bernoulli beam with Kelvin-Voigt or strain rate damping, the internal moment is composed of two components representing resistance to bending (with coefficient $EI(x)$) and damping (with coefficient $c_D I(x)$):

$$M(t,x) = EI(x)\frac{\partial^2 y}{\partial x^2}(t,x) + c_D I(x)\frac{\partial^3 y}{\partial x^2 \partial t}(t,x). \tag{2}$$

equ:1.2

If piezoelectric elements are bonded to the beam in a configuration to produce (or sense) only bending, we have an actuator contribution $M_p(t, x)$ in the form of an input moment (or voltage output proportional to the strain in the beam). For a pair of piezoelectric actuators located between x_1 and x_2 on opposite sides of the beam excited by a voltage $u(t)$ in an out-of-phase manner (see [?, ?, ?, ?, ?]), this moment term has the representation

$$M_p(t,x) = K_B\{H(x-x_1) - H(x-x_2)\}\,u(t) \tag{3}$$

equ:1.3

where $H(x)$ is the Heaviside or step function and K_B is a piezoelectric material parameter depending on the material piezoelectric properties as well as geometry. When the moment in ?? is added to that of ?? and substituted into ??, we obtain the model

$$\begin{aligned} \rho\frac{\partial^2 y}{\partial t^2} + \gamma\frac{\partial y}{\partial t} + \frac{\partial^2}{\partial x^2}\Big(EI\frac{\partial^2 y}{\partial x^2} + c_D I\frac{\partial^3 y}{\partial x^2\partial t}\Big) \\ = K_B\left(\frac{d}{dx}\delta(x-x_2) - \frac{d}{dx}\delta(x-x_1)\right)u(t) \\ y(t,0) = \frac{\partial y}{\partial x}(t,0) = 0, \quad M(t,\ell) = \frac{\partial M}{\partial x}(t,\ell) = 0 \end{aligned} \tag{4}$$

equ:1.4

where δ is the Dirac delta function. This is equivalent to the equation in weak or variational form (we replace partial derivatives in time by subscript t and space by superscript $'$)

$$\begin{aligned} &< \rho y_{tt} + \gamma y_t, \phi > + < EI\,y'' + c_D I\,y_t'' + K_B(H_1 - H_2)\,u(t), \phi'' >= 0 \\ &y(t,0) = y'(t,0) = 0, \end{aligned} \tag{5}$$

equ:1.5

for all $\phi \in H^2(0,\ell)$ satisfying $\phi(0) = \phi'(0) = 0$. Here H_i is the shifted Heaviside function $H_i(x) = H(x - x_i)$, $i = 1,2$ and $< \cdot, \cdot >$ is the usual L_2 inner product.

For the same configuration, when the beam is under deformation (bending), the generated charges in terms of voltage across the piezoelectric sensors has the expression (see [?])

$$K_s\int_{x_1}^{x_2}\frac{\partial^2 y}{\partial x^2}(t,x)dx = K_s\Big(\frac{\partial y}{\partial x}(t,x_2) - \frac{\partial y}{\partial x}(t,x_1)\Big)$$

where K_s is a sensor constant which is also a piezoelectric material properties and geometry related quantity.

The system ?? is a formal representation of the dynamics of a damped beam with piezoelectric actuators. To develop computational techniques (e.g., finite elements) based on rigorous convergence arguments, it is necessary to first have a precise formulation of this system. In Section 2 below we give general well-posedness results (existence and uniqueness of solution) for these equations. Since our interest is in computationally tractable methods for identification of variable coefficient distributed parameters and since the systems are infinite dimensional state systems, approximation methods are necessary. We formulate approximation techniques in the spirit of Galerkin techniques. In Section 3 we present convergence and continuous dependence in the context of parameter estimation problems. Examples for our investigations and numerical findings are given in Section 4 together with important findings related to comparison with experimental data. We also present important conclusions that one can draw from our efforts.

II. The Existence and Uniqueness of the Solutions

In this section we present an abstract formulation for second order systems in variational form. The beam with piezoelectric actuator of Section 1 is a special case of these systems. Let V, V_2 and H be complex Hilbert spaces such that $V \hookrightarrow V_2 \hookrightarrow H \hookrightarrow V_2^* \hookrightarrow V^*$ (an extended Gelfand triple construction–see [?]). The spaces V^* and V_2^* are the topological dual spaces of V and V_2, respectively. As usual, we identify the Hilbert space H with H^*.

The general second order system we consider is given by

$$\begin{aligned} &< \ddot{y}(t), \psi >_{V^*,V} + \sigma_1(y(t), \psi) + \sigma_2(\dot{y}(t), \psi) = < f(t), \psi >_{V_2^*,V_2} \quad \text{for } \psi \in V, \\ &y(0) = y_0, \quad \dot{y}(0) = y_1. \end{aligned} \tag{6}$$ equu:2.1

Here we use $< \ , \ >_{V^*,V}$ to denote the usual [?] duality product obtained as the extension by continuity of the H-inner product from $H \times V$ to $V^* \times V$. The term σ_1 is a sesquilinear form on V satisfying V-ellipticity and V-continuity conditions and σ_2 is a sesquilinear form on V_2 satisfying V_2-ellipticity and V_2-continuity conditions. That is, we assume that σ_1 and σ_2 satisfy

$$Re\, \sigma_1(\phi, \phi) \geq k_1 |\phi|_V, \qquad |\sigma_1(\phi, \psi)| \leq c_1 |\phi|_V \cdot |\psi|_V \tag{7}$$ equ:2.1a

$$Re\, \sigma_2(\xi, \xi) \geq k_2 |\xi|_{V_2}, \qquad |\sigma_2(\xi, \eta)| \leq c_2 |\xi|_{V_2} \cdot |\eta|_{V_2} \tag{8}$$ equ:2.1b

for $k_1, \ k_2 > 0, \ \phi, \psi \in V$ and $\xi, \eta \in V_2$. Under weak assumptions on f, the system **??** has a unique solution.

thm:2.1 **Theorem 1** *If the sesquilinear forms σ_1 and σ_2 satisfy conditions* **??** *and* **??** *with σ_1 symmetric and $f \in L_2((0,T), V_2^*)$, then, for each $w_0 = (y_0, y_1) \in \mathcal{H} = V \times H$, the initial value problem* **??** *has a unique solution $w(t) = (y(t), \dot{y}(t)) \in L_2((0,T), V \times V_2)$. Moreover, this solution depends continuously on f and w_0 in the sense that the mapping $\{w_0, f\} \to w = (y, \dot{y})$ is continuous from $\mathcal{H} \times L_2((0,T), V_2^*)$ to $L_2((0,T), V \times V_2)$.*

For a detailed proof of this theorem see [?]. In the above general formulation, we posed a weak assumption on the damping form σ_2. Such a weak assumption is useful in treating forms of internal damping that are weaker than the Kelvin-Voigt damping of Section 1; see [?] for examples. In [?], it is also established that one can formulate an extended semigroup theory for solutions of ?? given in terms of a variation of parameters representation and that this solution is the same as that guaranteed by Theorem 1.

In our example of Section 1 for a cantilevered beam with Kelvin-Voigt damping, we have the Hilbert spaces $V_2 = V$ and the sesquilinear forms

$$\sigma_1(y,\phi) = < EI\, y'', \phi'' >_H \tag{9}$$ equ:2.5

$$\sigma_2(\dot{y},\phi) = < c_D I\, \dot{y}'', \phi'' >_H \qquad \text{for } \phi \in V \tag{10}$$ equ:2.6

with the spaces defined by $V = H_L^2(0,\ell) = \{\phi \in H^2(0,\ell) \,|\, \phi(0) = \phi'(0) = 0\}$ and $H = L_2(0,\ell)$. The term $f(t)$ is given by

$$f(t,x) = K_B \cdot (H''(x-x_1) - H''(x-x_2)) \cdot u(t), \qquad 0 \le x, x_1, x_2 \le \ell \tag{11}$$ equ:2.7

where again $H(x)$ is the Heaviside function and $f(t,x)$ belongs to the dual space (see [?]) $V^* = \left(H_L^2(0,\ell)\right)^*$. If both $|EI(x)|_{L_\infty} \ge \alpha$ and $|c_D I(x)|_{L_\infty} \ge \alpha$ for some $\alpha > 0$, then σ_1 and σ_2 are V-elliptic and continuous with σ_1 symmetric; hence by Theorem ?? our beam equation is well posed for $f(t,x)$ given by ??.

If in place of the cantilevered beam of Section 1 we consider a beam with fixed ends (i.e. we have $y = \frac{\partial y}{\partial x} = 0$ at both $x = 0$ and $x = \ell$), then the space V is given by the usual Sobolov space $V = H_0^2(0,\ell)$ and $V^* = H^{-2}(0,\ell)$. If one again considers Kelvin-Voigt damping we choose $V_2 = V = H_0^2(0,\ell)$ whereas a choice of the so-called square root damping $\sigma_2(\dot{y},\phi) = < c_D I \dot{y}', \phi' >_H$ leads to $V_2 = H_0^1(0,\ell)$, $V_2^* = H^{-1}(0,\ell)$, and viscous or air damping leads to $V_2 = H = L_2(0,\ell)$ and $V_2^* = H^* = H$.

III. Parameter Estimation and Approximation

The parameter estimation problems which we consider for the beam with piezoelectric actuators and sensors can be stated in terms of finding parameters which give the best fit of the parameter dependent solutions of the partial differential equations to the observation data for response of the system to various excitations. In our case, the parameters to be estimated include beam mass density $\rho(x)$, stiffness coefficient $EI(x)$ as well as damping parameters $c_D I(x)$, γ and piezoelectric material parameters K_B, K_s. Let the collection of unknown parameters be denoted by $q = (\rho(x), EI(x), c_D I(x), \gamma, K_B, K_s)$. We then can consider the least squares estimation problem of minimizing over $q \in Q$ the least squares functional

$$J(q) = \sum_i \left| \mathcal{C} y(t_i; q) - z_i \right|^2, \tag{12}$$ equ:3.0

where $\{z_i\}$ are given observations and $\{y(t_i;q)\}$ are the parameter dependent weak solutions of **??** or **??** evaluated at each time t_i, $i = 1, 2, \cdots, \bar{N}$. The set Q is some admissible parameter set while the operator $\mathcal{C}$ has forms depending on the type of sensors. When collected data are displacement, velocity, or acceleration at a point $\bar{x}$ (or several points) on the beam, we minimize

$$J_\nu(q) = \sum_i \left| \frac{\partial^\nu y}{\partial t^\nu}(t_i, \bar{x}; q) - z_i \right|^2. \qquad (13)$$

equ:3.0a

for $\nu = 0, 1, 2$, respectively. In this case the operator $\mathcal{C}$ involves differentiation (either $\nu = 0, 1$ or 2 times, respectively) with respect to time followed by pointwise evaluation in t and x. When a piezoelectric sensor is used, the functional to be minimized is

$$J_p(q) = \sum_i \left| K_s \Big(\frac{\partial y}{\partial x}(t_i, x_2; q) - \frac{\partial y}{\partial x}(t_i, x_1; q) \Big) - z_i \right|^2, \qquad (14)$$

equ:3.0b

for the piezoelectric elements being located on the beam between x_1 and x_2. Here $\{z_i\}$ are the measured voltages across the piezoelectric elements.

The minimization in our parameter estimation problems involves an infinite dimensional state and an infinite dimensional (functions) admissible parameter space. Motivated by computational requirements, we thus consider Galerkin type approximations in the context of the variational formulation of Section 2. Let H^N be a sequence of finite dimensional subspaces of H, and Q^M be a sequence of finite dimensional sets approximating the parameter set Q. We define the orthogonal projections $P^N : H \to H^N$ of H onto H^N. Then a family of approximating estimation problems with finite dimensional state spaces and parameter sets can be formulated by seeking $q \in Q^M$ which minimizes

$$J_\nu^N(q) = \sum_i \left| \frac{\partial^\nu y^N}{\partial t^\nu}(t_i, \bar{x}; q) - z_i \right|^2, \qquad \nu = 0, 1, 2, \qquad (15)$$

equ:3.1

or, in the event piezoelectric sensors are employed,

$$J_p^N(q) = \sum_i \left| K_s \Big(\frac{\partial y^N}{\partial x}(t_i, x_2; q) - \frac{\partial y^N}{\partial x}(t_i, x_1; q) \Big) - z_i \right|^2, \qquad (16)$$

equ:3.1a

where $y^N(t;q) \in H^N$ is the solution to the finite dimensional approximation of **??** given by

$$\begin{aligned} &< \ddot{y}^N(t), \psi >_{V^*,V} + \sigma_1(y^N(t), \psi) + \sigma_2(\dot{y}^N(t), \psi) =< f(t), \psi >_{V_2^*,V_2} \quad \text{for } \psi \in H^N, \\ &y^N(0) = P^N y_0, \quad \dot{y}^N(0) = P^N y_1. \end{aligned} \qquad (17)$$

equ:3.2

For the parameter sets Q and Q^M, and state spaces H^N, we make the following hypotheses:

(H1) The sets Q and Q^M lie in a metric space $\tilde{Q}$ with metric d with Q, Q^M compact in this metric and there is a mapping $i^M : Q \to Q^M$ so that $Q^M = i^M(Q)$. Furthermore, for each $q \in Q, i^M(q) \to q$ in $\tilde{Q}$ with the convergence uniform in $q \in Q$;

(H2) The finite dimensional subspaces H^N satisfy $H^N \subset V$ as well as the approximation properties of the next two statements;

(H3) For each $\psi \in V$, $|\psi - P^N\psi|_V \to 0 \qquad \text{as } N \to \infty$;

(H4) For each $\psi \in V_2$, $|\psi - P^N\psi|_{V_2} \to 0 \qquad \text{as } N \to \infty$.

We note that the sesquilinear forms introduced in the previous section are parameter dependent. In addition to (uniform in Q) ellipticity and continuity conditions **??** and **??**, the sesquilinear forms $\sigma_1 = \sigma_1(q)$ and $\sigma_2 = \sigma_2(q)$ are assumed to be defined on Q and satisfy the continuity-with-respect-to-parameter conditions

$$|\,\sigma_1(q)(\phi,\psi) - \sigma_1(\tilde{q})(\phi,\psi)\,| \le \gamma_1\, d(q,\tilde{q})\,|\phi|_V\,|\psi|_V, \qquad \text{for } \phi,\psi \in V \qquad (18)$$ equ:3.2a

$$|\,\sigma_2(q)(\xi,\eta) - \sigma_2(\tilde{q})(\xi,\eta)\,| \le \gamma_2\, d(q,\tilde{q})\,|\xi|_{V_2}\,|\eta|_{V_2}, \qquad \text{for } \xi,\eta \in V_2 \qquad (19)$$ equ:3.2b

for $q, \tilde{q} \in Q$ where the constants $\gamma_1,\ \gamma_2$ depend only on Q.

We are now able to state an important result related to the convergence and continuous dependence (with respect to data) of the solutions to the approximate optimization problems involving **??** or **??** and **??**. Indeed, the results of the next theorem allow one to conclude in many instances that solutions of the approximate problems converge (in an appropriate sense) to solutions for the original estimation problems involving **??** or **??** and **??**. (See the discussions below and [**?**, **?**] for more details.)

thm:3.1 **Theorem 2** *Suppose that H^N satisfies (H2), (H3), (H4). Assume the sesquilinear forms $\sigma_1(q)$ and $\sigma_2(q)$ satisfy **??**, **??**, **??** and **??**. Furthermore, assume that*

$$q \to f(t;q) \textit{ is continuous from } Q \textit{ to } L_2((0,T),V_2^*). \qquad (20)$$ equ:3.3a

Let q^N be arbitrary in Q such that $q^N \to q$ in Q; then for $t > 0$ as $N \to \infty$ we have

$$y^N(t;q^N) \to y(t;q) \qquad \textit{in } V \textit{ norm},$$

$$\dot{y}^N(t;q^N) \to \dot{y}(t;q) \qquad \textit{in } V_2 \textit{ norm},$$

*where y^N, $\dot{y}^N$ are the solutions to **??** and y, $\dot{y}$ are the solutions to **??**.*

Proof: We recall that the solution of ?? satisfies $(y(t), \dot{y}(t)) \in V \times V_2$. Since

$$|y^N(t;q^N) - y(t;q)|_V \le |y^N(t;q^N) - P^N y(t;q)|_V + |P^N y(t;q) - y(t;q)|_V,$$

and (H3) implies that second term on the right side converges to 0 as $N \to \infty$, it suffices for the first convergence statement to show that

$$|y^N(t;q^N) - P^N y(t;q)|_V \to 0 \qquad \text{as N} \to \infty.$$

Similarly, we note that this same inequality with y^N, y replaced by $\dot{y}^N$, $\dot{y}$ and the V-norm replaced by the V_2-norm along with (H4) permits us to claim that the convergence

$$|\dot{y}^N(t;q^N) - P^N \dot{y}(t;q)|_{V_2} \to 0 \qquad \text{as N} \to \infty,$$

is sufficient to establish the second convergence statement of the theorem. We shall, in fact, establish the convergence of $\dot{y}^N - P^N\dot{y}$ in the stronger V norm.

Let $y^N = y^N(t;q^N)$, $y = y(t;q)$, and $z^N \equiv y^N(t;q^N) - P^N y(t;q)$. Then

$$\dot{z}^N = \dot{y}^N - \frac{d}{dt}P^N y = \dot{y}^N - P^N\dot{y}$$

and

$$\ddot{z}^N = \ddot{y}^N - \frac{d^2}{dt^2}P^N y$$

since $\dot{y} \in L_2((0,T),V_2)$, $\ddot{y} \in L_2((0,T),V^*)$. From ?? and ??, we have for $\psi \in H^N$

$$\begin{aligned} < \ddot{z}^N, \psi >_{V^*,V} &= < \ddot{y}^N - \ddot{y} + \ddot{y} - \frac{d^2}{dt^2}P^N y, \psi >_{V^*,V} \\ &= < f(q^N), \psi >_{V_2^*,V_2} - \sigma_2(q^N)(\dot{y}^N,\psi) - \sigma_1(q^N)(y^N,\psi) \\ &\quad - < f(q), \psi >_{V_2^*,V_2} + \sigma_2(q)(\dot{y},\psi) + \sigma_1(q)(y,\psi) \\ &\quad + < \ddot{y} - \frac{d^2}{dt^2}P^N y, \psi >_{V^*,V} . \end{aligned} \tag{21}$$

equ:thm1

This can be written as

$$\begin{aligned} &< \ddot{z}^N, \psi >_{V^*,V} + \sigma_1(q^N)(z^N,\psi) \\ &\quad = < \ddot{y} - \frac{d^2}{dt^2}P^N y, \psi >_{V^*,V} - < f(q) - f(q^N), \psi >_{V_2^*,V_2} \\ &\quad\; + \sigma_2(q^N)(\dot{y} - P^N\dot{y},\psi) + \sigma_2(q)(\dot{y},\psi) - \sigma_2(q^N)(\dot{y},\psi) \\ &\quad\; + \sigma_1(q^N)(y - P^N y,\psi) + \sigma_1(q)(y,\psi) - \sigma_1(q^N)(y,\psi) \\ &\quad\; - \sigma_2(q^N)(\dot{z}^N,\psi). \end{aligned} \tag{22}$$

equ:thm2

Choosing $\dot{z}^N$ as the test function ψ in ?? and using the equality $< \ddot{z}^N, \dot{z}^N >_{V^*,V} = \frac{1}{2}\frac{d}{dt}|\dot{z}^N|_V^2$ (this follows using definitions of the duality mapping–see [?] and the hypothesis (H2)), we have (here we also use symmetry of σ_1)

$$\frac{1}{2}\frac{d}{dt}\{\ |\dot{z}^N|_V^2 + \sigma_1(q^N)(z^N, z^N)\ \}$$
$$\begin{aligned} = \ & Re\Big\{ < \ddot{y} - \frac{d^2}{dt^2}P^N y, \dot{z}^N >_{V^*,V} - < f(q) - f(q^N), \dot{z}^N >_{V_2^*,V_2} \\ & +\sigma_2(q^N)(\dot{y} - P^N\dot{y}, \dot{z}^N) + \sigma_2(q)(\dot{y}, \dot{z}^N) - \sigma_2(q^N)(\dot{y}, \dot{z}^N) \\ & +\sigma_1(q^N)(y - P^N y, \dot{z}^N) + \sigma_1(q)(y, \dot{z}^N) - \sigma_1(q^N)(y, \dot{z}^N) \\ & -\sigma_2(q^N)(\dot{z}^N, \dot{z}^N)\Big\}. \end{aligned} \tag{23}$$

equ:thm3

Integrating the terms in **??** from 0 to t, applying **??**, **??**, **??**, **??**, and the initial conditions

$$z^N(0) = y^N(0) - P^N y(0) = y^N(0) - P^N y_0 = 0$$
$$\dot{z}^N(0) = \dot{y}(0) - P^N\dot{y}(0) = \dot{y}(0) - P^N y_1 = 0,$$

we obtain

$$|\dot{z}^N|_V^2 + k_1|z^N|_V^2 \le \nu_1\Delta^N(t) + \nu_2\int_0^t \{|\dot{z}^N|_V^2 + k_1|z^N|_V^2\}\,ds, \tag{24}$$

equ:thm4

where ν_1, ν_2 and k_1 are positive constants not dependent on N, and

$$\begin{aligned} \Delta^N = \int_0^t \ & \Big\{ < \ddot{y} - \frac{d^2}{dt^2}P^N y, \dot{z}^N >_{V^*,V} + |f(q) - f(q^N)|_{V_2^*}^2 + c_2^2\,|\dot{y} - P^N\dot{y}|_{V_2}^2 \\ & + \gamma_2^2\, d^2(q, q^N)\,|\dot{y}|_{V_2}^2 + c_1^2\,|y - P^N y|_V^2 + \gamma_1^2\, d^2(q, q^N)\,|y|_V^2 \Big\}\, ds. \end{aligned} \tag{25}$$

equ:thm5

We claim that for $q^N \to q \in Q$, the term Δ^N converges to 0 as $N \to \infty$. To establish this, we first note that $< \ddot{y} - \frac{d^2}{dt^2}P^N y, \dot{z}^N >_{V^*,V} \equiv 0$. Indeed, for any $\psi \in H^N$, we have

$$\begin{aligned} < \ddot{y} - \frac{d^2}{dt^2}P^N y, \psi >_{V^*,V} \ &= \ < \frac{d^2}{dt^2}(y - P^N y), \psi >_{V^*,V} \\ &= \frac{d^2}{dt^2} < y - P^N y, \psi >_{V^*,V} \\ &= \frac{d^2}{dt^2} < y - P^N y, \psi >_H . \end{aligned} \tag{26}$$

equ:orth

But $< y - P^N y, \psi >_H \equiv 0$ since $(y - P^N y)$ is orthogonal to elements in H^N. We note that the last equality in **??** follows from the fact that the duality pairing $< \cdot, \cdot >_{V^*,V}$ is the extension by continuity of the inner product $< \cdot, \cdot >_H$ from $H \times V$ to $V^* \times V$ and hence for $h \in V$, $< g, h >_{V^*,V} = < g, h >_H$ whenever $g \in H = H^*$. The remainder of the terms in Δ^N approach zero as $N \to \infty$ due to (H3)–(H4), the continuity condition **??**, and $q^N \to q$.

Applying Gronwall's inequality to **??**, we obtain

$$|\dot{z}^N|_V^2 + k_1|z^N|_V^2 \to 0$$

as $N \to \infty$, and hence the convergence statement of the theorem.

As we have noted, the results of Theorem 2 can be used to establish (subsequential) convergence of solutions $\bar{q}^{N,M}$ of the approximation problems of minimizing J_ν^N, $\nu = 0, 1$, or J_p^N of ?? or ?? and ?? over Q^M to solutions $\bar{q}$ of the original (infinite dimensional) problems of minimizing J_ν or J_p of ?? or ?? and ?? over Q. Indeed, under (H1)–(H4) one can establish results (see [?] and [?, p. 61-65] for more complete details and discussions) in the more general sense of the ideas of "method stability" introduced in [?].

The case of minimizing J_ν^N and J_ν for $\nu = 2$ (i.e. accelerometer data) is slightly more delicate. In the case that $V_2 = V$ (strong damping such as the case of Kelvin-Voigt damping in the beam example of Section 1) one can use ideas from analytic semigroup theory (see similar examples in [?]) to establish an analogue of Theorem 2 involving convergence of $\ddot{y}^N(t; q^N)$ to $\ddot{y}(t; q)$, thereby obtaining a desired convergence of solutions for the corresponding minimization problems. In the case that $V_2 \neq V$, we do not have a general theory for convergence in the case of accelerometer data.

We return to the cantilevered beam example of Section 1 and argue that all the assumptions for Theorem 2 can be satisfied for this example. First, we note that the sesquilinear forms defined by ?? and ?? satisfy ?? and ?? while f defined by ?? satisfies the required continuity condition ?? if we choose $H = L_2(0, \ell)$, $V = V_2 = H_L^2(0, \ell)$ and $q = (\rho, EI, c_D I, \gamma, K_B, K_s)$ in Q, a compact subset of $\tilde{Q} = [L_\infty(0, \ell)]^3 \times \mathbb{R}^2$. Moreover, $y^N(t; q^N) \to y(t; q)$ in this V norm implies $Dy^N(t; q^N) \to Dy(t; q)$ in the H^1 norm, hence we have the pointwise convergence required if J_p of ?? is used as the fit criterion. If we choose cubic splines for the basis of the approximation scheme, and the parameter set Q as a uniformly bounded collection of piecewise constant functions, then (H1)–(H4) are satisfied (verification for (H3) and (H4) will be given shortly) and the desired convergence of Theorem ?? will be achieved.

With the chosen Q and $\tilde{Q}$, i^M is taken as the identity so that (H1) is met. To argue (H2)–(H4), we construct H^N as following. On the spatial interval $\Omega = [0, \ell]$, corresponding to the equidistant partition $\{\frac{i\ell}{N}\}_{i=0}^N$ consider the family given by

$$S_{3,B}^N(0,\ell) = \Big\{p \in C^2(0,\ell) : p \text{ is a cubic polynomial on each subinterval } [\frac{i\ell}{N}, \frac{(i+1)\ell}{N}], \quad 0 \le i \le N\Big\}$$

and define H^N as the span of the basis set for $S_{3,B}^N(0,\ell)$ (see [?]) with the basis set modified to satisfy essential boundary conditions $\phi(0) = D\phi(0) = 0$. It is obvious that $H^N \subset V$.

We conclude this section with a theorem which guarantees the hypothesis (H3) (and also (H4) in this case since $V = V_2$).

thm:3.2 **Theorem 3** *Let P^N be the orthogonal projection of $L^2(0,\ell)$ onto $S^N_{3,N}(0,\ell)$ and let $\psi \in H^2(0,\ell)$. Then there exist constants α_1 and α_2 independent of N and ψ such that*

$$\begin{aligned} |\psi - P^N\psi|_{L^2} &\le \alpha_1 N^{-2} |D\psi|_{L^2} \\ |D(\psi - P^N\psi)|_{L^2} &\le \alpha_2 N^{-1} |D\psi|_{L^2}. \end{aligned}$$

Furthermore, $|D^2(\psi - P^N\psi)|_{L^2} \to 0$ as $N \to \infty$.

Proof: The proof employs arguments very similar to those in the proof of Theorem A.5.4 in [?]. For completeness, we give a brief outline here.

From [?], for $\psi \in H^2(0,\ell)$ we have the following estimates (we denote the L_2-norm by $|\cdot|$)

$$|\psi - I^N_Q\psi| \le 4c_1 N^{-2} |D^2\psi|, \tag{27}$$ 3.8

$$|D(\psi - I^N_Q\psi)| \le 4c_1 N^{-1} |D^2\psi|, \tag{28}$$ 3.9

$$|DI^N_Q\psi| \le 4c_1 |D^2\psi| \tag{29}$$ 3.10

where I^N_Q is the quasi-interpolation operator (e.g. see Theorem A.5.2 of [?] for details) corresponding to N^{-1}, and c_1 is a fixed constant independent of N and ψ. The first estimate of the theorem follows (with $\alpha_1 = \alpha_2 = 4c_1$) from **??** and the minimizing property of the projection P^N, $|\psi - P^N\psi| \le |\psi - I^N_Q\psi|$. The second estimate follows from **??**–**??** along with use of the Schmidt inequality.

From Theorem A.1.2 of [?], the Schmidt inequality and **??**, we have for $h \in H^2(0,\ell)$

$$|D^2P^Nh| \le K\,|D^2h|$$

with $K = 4c_1\,(2\tilde{C} + 1)$, where $\tilde{C}$ is independent of h and N and can be calculated explicitly. Then from density arguments (e.g. take $g \in C^4$) and the inequality

$$\begin{aligned} |D^2(\psi - P^N\psi)| &\le |D^2(\psi - g)| + |D^2(g - P^Ng)| + |D^2P^N(g - \psi)| \\ &\le (1 + K)\,|D^2(\psi - g)| + |D^2(g - P^Ng)|, \end{aligned}$$

the convergence statement of the theorem follows from the standard estimates of Theorem A.4.2 of [?] and use of the Schmidt inequality again (see p. 79 of [?]).

We remark that for cases of weaker (than Kelvin-Voigt) damping, i.e. $V_2 \neq V$, we still have (H3) and (H4) holding as long as $V_2 = H^1_L(0,\ell)$ (e.g., see Theorem A.5.4 of [?]).

IV. Numerical and Experimental Results

To test the above described estimation theory and computational procedures, a series of experiments were carried out at the Mechanical Systems Laboratory, then located at the State University of New York at Buffalo (now at Virginia Polytechnic Institute and State University). A cantilevered aluminum beam with two attached piezoceramic patches was used as the test structure. The patches were bonded with Epoxy Adhesive to a aluminum beam on the opposite sides of the beam at the same position. In the following two tables, the subscripts indicate the materials: b for beam and p for piezoceramic. Let ℓ be length, w be width and t be thickness; then directly measured dimensions and the handbook values of the physical characteristics (stiffness and mass density) for a 2024-T4 aluminum beam are:

Table 1: Experimental beam dimensions and its characteristics.

ℓ_b (cm)	w_b (cm)	t_b (cm)	E_b (N/cm^2)	ρ_b (g/cm)
45.73	2.03	0.16	7.3×10^6	0.89

For a Piezoelectic Products model G-1195 PZT ceramic, the book values of the characteristics and the dimensions of the patch used are:

Table 2: PZT ceramic patch dimensions and its characteristics.

ℓ_p (cm)	w_p (cm)	t_p (cm)	E_p (N/cm^2)	ρ_p (g/cm)
6.37	2.03	0.0254	6.3×10^6	0.78

In the tables, E is the Young's modulus and ρ is the mass density per unit length.

The beam was clamped at $x = 0$. The center of the piezoceramic patch was placed at 5.72 cm away from the clamped end. One 0.64 cm wide and 0.0076 cm thick copper foil to act as conducting media was glued on the beam under each piezoceramic patch. The time response data and input signal from the experimental beam were obtained using the Tektronix Analyzer (model 2600).

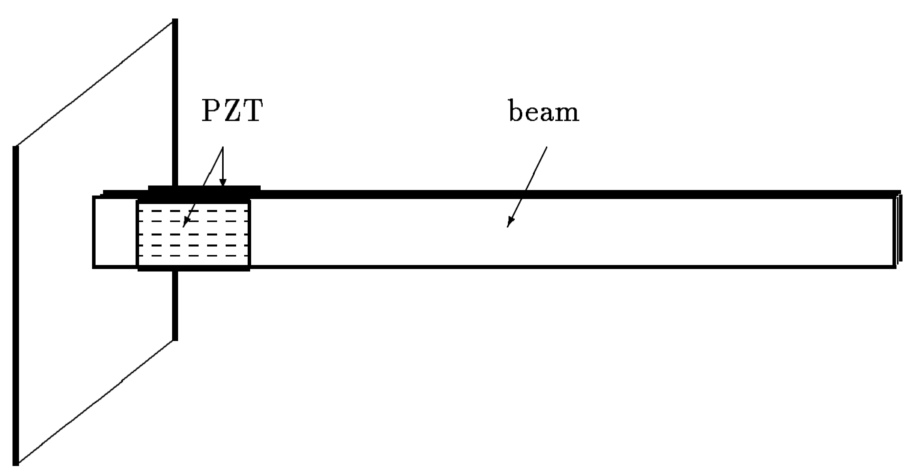
In the first example, labeled B1TAPIH, the piezoelectric ceramic patches were used as a sensor; that is, the voltage across the patch due to the beam vibration was observed. The beam was excited by an impulse force applied (via an impulse hammer) on the beam along its neutral axis at 2.54 cm away from the clamped end. The input signal was recorded from the transducer hammer (the actuator).

At a sampling frequency of 256 Hz, two modes were observed in the response, at 6.625 Hz and 38.375 Hz, respectively.

Since the piezoceramic patches were not used as an actuator, the input voltage to the patches was zero. Hence the term

$$K_B \left(\delta'(x - x_2) - \delta'(x - x_1) \right) u(t)$$

in equation **??** is zero. Instead, a term $f(t,x)$ which represents the hammer input is introduced on the right hand side of the partial differential equation in **??**. The functional J_p in equation **??** was minimized in attempts to fit the data. In this example, K_B was not involved, and therefore was set to zero. A key feature of the proposed method is that the mass density, stiffness and Kelvin-Voigt damping coefficients are functions of position along the beam. To agree with geometry of the structure, we assumed that they are piecewise constant functions as shown in Figure 1. Such assumptions were later verified to be reasonable (indeed necessary in some respects).



In the second example, labeled B1TAOIV, the piezoceramic patches were used as the actuator and an accelerometer was used as the sensor. The accelerometer weighing 0.5 gram was located at $x = 2.14$ cm. Our choice of the location of the accelerometer was to made so as to minimize the dynamic effects of the accelerometer (e.g. effects due to the weight of the accelerometer and to the wire attached to it). A narrow triangle (approximating an impulse) voltage was applied to the patches to excite the beam. The ceramic patches were excited out of phase so as to produce input moments as modeled in **??** or **??** above. In order to maintain a constant E_p through out the data acquisition period following excitation when only accelerometer data was collected (i.e. the ceramic patch was not used as a sensor), a zero voltage supply (not zero current) to the patches was provided.

In both cases, for computations the dimension of the approximation space N was set to 10 since the eigenvalues of the approximate finite dimensional system became stable at and after $N = 10$ in the sense that the eigenvalues do not change significantly as N increases beyond 10.

In the first example, we began the parameter identification scheme by holding damping related parameters fixed while identifying parameters $EI(x)$ and $\rho(x)$ to first obtain a match of frequencies. We used measured values together with book values as our initial guess. The initial values shown in Table 3 as "given" values were calculated according to the following equations

$$EI(x) = \frac{1}{12}t_b^3 w_b E_b + \frac{2}{3}\left[\left(\frac{t_b}{2}+t_p\right)^3 - \left(\frac{t_b}{2}\right)^3\right] w_p E_p \chi_p(x) \qquad (30)$$ equ:4.1

$$\rho(x) = \rho_b + \rho_p \chi_p(x) \qquad (31)$$ equ:4.2

where χ_p is a characteristic function given by

$$\chi_p(x) = \begin{cases} 1 & x_1 \le x \le x_2 \\ 0 & \text{otherwise.} \end{cases}$$

The derivation of equation **??** can be found in [?]. Equation **??** is simply the superposition of linear mass density of the beam and the patches. Notice that in the above evaluations, the glue and copper foil were ignored. Then the estimation was carried out on the damping parameters $c_D I(x)$ and γ, and piezoceramic related parameter K_s, while keeping the parameters $EI(x)$ and $\rho(x)$ at the optimal values obtained. The initial values were $c_D I(x) = 0.825 \times 10^{-5}$ and $\gamma = 0.00183$. The optimal values obtained from the first example were used as initial values in the second example. Since both examples are from the same structure with different sensors and actuators, we anticipated that the estimated parameters from the two examples might be close. This agreement is offered as verification of the methodology proposed here as it adds a credible element of consistency not found in previous work.

A summary of the estimation results is given in Table 3. For comparison, results from both examples are listed in the same table. The measured and handbook quantities (when available) are also listed in the table as "given" values.

Table 3: Given and estimated structural parameters

		given	B1TAPIH	B1TAOIV
EI	beam	0.495	0.491	0.505
(N·m^2)	beam+PZT	1.050	0.793	0.798
ρ	beam	0.089	0.093	0.096
(kg/m)	beam+PZT	0.168	0.433	0.441
$c_D I$	beam	—	0.649×10^{-5}	0.637×10^{-5}
(s·N·m^2)	beam+PZT	—	1.255×10^{-5}	1.275×10^{-5}
γ (s·N/m^2)		—	0.013	0.013
$\tilde{K}_s$ (v)		—	4682.342	—
K_B (N·m/v)		—	—	1.746×10^{-2}

The results (graphs) are reported in the order described above. The example B1TAPIH is given in Figure 2 and the example B1TAOIV is shown in Figure 3. In each figure, there are four parts: (a) is the recorded experimental data, (b) is the model response with the estimated parameters given in Table 3, (c) is the amplitude of the FFT of the experimental data (in solid lines) and model response (in dashed lines), and in part (d), both experimental data (in solid lines) and the model response (in dashed lines) are presented on a shorter time interval in one plot to exhibit the details of how well the model fits the experimental data.