Computational Methods for Identification and Feedback Control in Structures with Piezoceramic Actuators and Sensors*

H.T. Banks,
K. Ito
Y. Wang

Center for Applied Mathematical Sciences
University of Southern California
Los Angeles, CA 90089-1113

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ABSTRACT

In this note we give fundamental existence, uniqueness, and continuous dependence results (well-posedness) for a variational formulation of a class of damped second order partial differential equations with unbounded input or control coefficients. Included as special cases in this class are structures with piezoceramic actuators. We then consider approximation techniques leading to computational methods in the context of both parameter estimation and feedback control problems for these systems. Rigorous convergence results for parameter estimates and feedback gains are presented.

1. INTRODUCTION

One of the important issues in design and understanding of smart structures and materials is the development of high fidelity dynamic models for use in identification and control methodologies. Along with these models, which are inherently distributed in nature, one requires computational techniques which must in turn be based on approximation ideas.

Our recent efforts with piezoceramic actuators and sensors in structural vibration suppression, acoustic noise suppression, and nondestructive evaluation of materials have prompted us to consider fundamental questions related to a mathematical framework for computational methods in distributed parameter systems for smart or adaptive structures. We discuss these ideas in the context of a cantilevered beam with piezoceramic patches for actuation and sensing. Current modeling practice leads to a beam equation with unbounded input (coefficients involve first derivatives of the Dirac delta function) resulting from excitation of the moment-producing actuator patches.

To be more specific, we consider an Euler-Bernoulli beam with Kelvin-Voigt internal damping and viscous (air) external damping. For a beam of length ℓ fixed at $x = 0$ and free at $x = \ell$, we find that the transverse vibrations are described by the

H.T. Banks, K. Ito and Y. Wang, Center for Applied Mathematical Sciences, University of Southern California, Los Angeles, CA 90089-1113

system for $y = y(t, x)$

$$\begin{aligned} &\rho \frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial y}{\partial t} + \frac{\partial^2 M}{\partial x^2} = g \qquad 0 < x < \ell,\ t > 0, \\ &y(t,0) = \frac{\partial y}{\partial x}(t,0) = 0, \quad M(t,\ell) = \frac{\partial M}{\partial x}(t,\ell) = 0. \end{aligned} \tag{1}$$

Here as usual y is the transverse displacement, ρ is the linear mass density, g is an external applied distributed force and M is the internal moment. For a simple Euler-Bernoulli beam with Kelvin-Voigt or strain rate damping, the internal moment is composed of two components representing resistance to bending and damping:

$$M(t,x) = EI \frac{\partial^2 y}{\partial x^2}(t,x) + c_D I \frac{\partial^3 y}{\partial x^2 \partial t}(t,x). \tag{2}$$

If piezoelectric actuators are attached to the beam in a configuration to produce only bending (patches on opposite sides of the beam excited in an out-of-phase manner – see [5, 6, 7, 8]), we have an actuator contribution $M_p(t,x)$ in the form of an input moment. For patches located between x_1 and x_2 on the beam excited out-of-phase by a voltage $u(t)$, this moment term has the representation

$$M_p(t,x) = K_B \{H(x - x_1) - H(x - x_2)\}\, u(t) \tag{3}$$

where H is the Heaviside or unit step at zero function and K_B is a piezoceramic material parameter depending on material properties of the beam and the patches as well as geometry. When the moment in (3) is added to that of (2) and substituted into (1), we obtain the model (throughout we assume that the external distributed force $g = 0$)

$$\begin{aligned} &\rho \frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial y}{\partial t} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2} + c_D I \frac{\partial^3 y}{\partial x^2 \partial t}\right) = K_B \left(\delta'(x - x_2) - \delta'(x - x_1)\right) u(t) \\ &y(t,0) = \frac{\partial y}{\partial x}(t,0) = 0, \quad M(t,\ell) = \frac{\partial M}{\partial x}(t,\ell) = 0 \end{aligned} \tag{4}$$

where δ is the Dirac delta function and $' = \frac{\partial}{\partial x}$. This is formally equivalent to the equation in weak or variational form

$$\begin{aligned} &< \rho y_{tt} + \gamma y_t, \phi > + < EI\, y'' + c_D I\, y_t'' + K_B (H_1 - H_2)\, u(t), \phi'' >= 0 \\ &y(t,0) = y'(t,0) = 0, \end{aligned} \tag{5}$$

for all sufficiently smooth ϕ satisfying $\phi(0) = \phi'(0) = 0$. Here H_i is the shifted Heaviside function $H_i(x) = H(x - x_i)$.

In this note we outline a mathematically rigorous setting for the systems (4) or (5). In section 2 below we give general well-posedness results (existence and uniqueness of solution) for these equations. Since our interest is in computationally tractable methods for identification of parameters and feedback control in such models and since the systems are infinite dimensional state systems, approximation methods are necessary. We formulate approximation techniques in the spirit of Galerkin techniques which include finite elements, spectral, and modal methods as special cases. In section 3 we present convergence results in the context of parameter estimation problems while in section 4 we outline results for convergence of feedback gains in a general linear quadratic regulator (LQR) problem.

2. ABSTRACT FORMULATION AND WELL POSEDNESS

In this section we present an abstract formulation for second order systems in variational form; these systems include the beam with piezoceramic actuator of section 1 as a special case. Let V and H be complex Hilbert spaces such that V is a dense subset of H and $|\cdot|_H \leq k|\cdot|_V$ for some constant k where $|\cdot|_H$ and $|\cdot|_V$ represent the H-norm and V-norm, respectively. We denote such a relationship between V and H by $V \hookrightarrow H$. Moreover, we identify the Hilbert space H with its topological dual H^*. It follows from $V \hookrightarrow H$ that $H^* \hookrightarrow V^*$ and thus we have the so-called Gelfand triple construction [19] denoted by $V \hookrightarrow H \hookrightarrow V^*$.

The general second order system we consider is given by

$$\begin{aligned} &< \ddot{y}(t), \psi >_{V^*,V} + \sigma_1(y(t),\psi) + \sigma_2(\dot{y}(t),\psi) = < f(t), \psi >_{V^*,V} \quad \text{for } \psi \in V, \\ &y(0) = y_0, \quad \dot{y}(0) = y_1. \end{aligned} \tag{6}$$

Here we use $< \, , \, >_{V^*,V}$ to denote the usual [19] duality product obtained as the extension by continuity of the H-inner product from $H \times V$ to $V^* \times V$, and σ_1 and σ_2 are sesquilinear forms on V satisfying V-ellipticity and continuity conditions. That is, we assume that σ_1 and σ_2 satisfy

$$Re\, \sigma_i(\phi,\phi) \geq k_i|\phi|_V, \qquad k_i > 0 \tag{7}$$

$$|\sigma_i(\phi,\psi)| \leq c_i|\phi|_V \cdot |\psi|_V \tag{8}$$

for $i = 1, 2$ and $\phi, \psi \in V$. The term $f(t)$ is the control or input term, i.e., $f(t) = Bu(t)$ as introduced in section 1. Under weak assumptions on f, the system (6) has a unique solution.

Theorem 1 *If the sesquilinear forms σ_1 and σ_2 satisfy conditions* (7) *and* (8) *with σ_1 symmetric and $f \in L_2((0,T),V^*)$, then, for each $w_0 = (y_0,y_1) \in \mathcal{H} = V \times H$, the initial value problem* (6) *has a unique solution $w(t) = (y(t),\dot{y}(t)) \in L_2((0,T),V \times V)$. Moreover, this solution depends continuously on f and w_0 in the sense that the mapping $\{w_0, f\} \to w = (y,\dot{y})$ is continuous from $\mathcal{H} \times L_2((0,T),V^*)$ to $L_2((0,T),V \times V)$.*

The solution of (6) is called the weak solution since this variational formulation is also often called a weak formulation. In actual fact, the existence and uniqueness of solutions of (6) can be obtained under even weaker assumptions on the damping form σ_2. Indeed, let V_2 be a complex Hilbert space satisfying $V \hookrightarrow V_2 \hookrightarrow H$ and suppose that σ_2 is a sesquilinear form on V_2. If σ_1 is V-elliptic and V-continuous and σ_2 is V_2-elliptic and V_2-continuous, then for $f \in L_2((0,T),V_2^*)$, there exists a unique solution $w(t) = (y(t),\dot{y}(t)) \in L_2((0,T),V \times V_2)$ to (6). Such a weak assumption on σ_2 is required to treat forms of internal damping that are weaker than the Kelvin-Voigt damping of section 1. For a proof of these results (which include Theorem 1 as a special case where $V_2 = V$) see [3] or [1].

We have in Theorem 1 stated the well posedness of the system (6) for $f \in L_2((0,T),V^*)$ in a weak variational setting. We can take an alternative (but as we shall see, equivalent) approach using the theory of semigroups ([9, 16]). Since the sesquilinear forms σ_1 and σ_2 are V-elliptic and continuous, we can define operators $A_i \in \mathcal{L}(V,V^*)$ in a standard manner by

$$\sigma_i(\phi,\psi) = < A_i\,\phi, \psi >_{V^*,V}, \qquad \text{for } \phi,\psi \in V, i = 1,2\,. \tag{9}$$

Here $\mathcal{L}(V,V^*)$ is the usual space of bounded linear operators from V to V^*. The corresponding abstract equation for (6) is then given by

$$\begin{aligned} &\ddot{y}(t) + A_2\dot{y}(t) + A_1 y(t) = f(t) \qquad \text{in } V^*, \\ &y(0) = y_0, \quad \dot{y}(0) = y_1\,. \end{aligned} \tag{10}$$

We will rewrite this second order system as a first order system for $w(t) = (y(t),\dot{y}(t))^T$ on a product space. We define the product space $\mathcal{V} = V \times V$ in addition to $\mathcal{H} = V \times H$ above. The first order system can be written as

$$\begin{aligned} &\dot{w}(t) = \mathcal{A}\,w(t) + F(t) \qquad \text{in } \mathcal{V}^*, \\ &w(0) = w_0, \end{aligned} \tag{11}$$

where $F(t) = (0, f(t))^T \in \mathcal{V}^*, w_0 = (y_0, y_1)^T \in \mathcal{H}$ and

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A_1 & -A_2 \end{pmatrix} \in \mathcal{L}(\mathcal{V},\mathcal{V}^*). \tag{12}$$

With the assumptions on σ_1 and σ_2 given in Theorem 1, the operator $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup $\mathcal{T}(t)$ on $\mathcal{V}^*$ (see [3] or [1]). Then, by definition, mild solutions of (11) in $\mathcal{V}^*$ are given by

$$w(t;q) = \mathcal{T}(t)\,w_0 + \int_0^t \mathcal{T}(t-s)\,F(s)\,ds\,. \tag{13}$$

For computational efforts in control and estimation of these systems, it is an important result to note that the weak formulation and the semigroup formulation yield the same solutions.

Theorem 2 *Suppose $w_0 = (y_0, y_1)^T \in \mathcal{H} = V \times H$, $f \in L_2((0,T),\mathcal{V}^*)$, and σ_1 and σ_2 are given as in Theorem 1. Then* (6) *has a unique solution in $L_2((0,T), V \times V)$ and it is given by the mild solution* (13).

In our example of section 1 for a cantilevered beam with Kelvin-Voigt damping, the sesquilinear forms are given by

$$\sigma_1(y,\psi) = < EI\,y'', \psi'' >_H \tag{14}$$
$$\sigma_2(\dot{y},\psi) = < c_D I\,\dot{y}'', \psi'' >_H \qquad \text{for } \psi \in V \tag{15}$$

with the spaces defined by $V = H_L^2(0,\ell) = \{\phi \in H^2(0,\ell) \,|\, \phi(0) = \phi'(0) = 0\}$ and $H = L_2(0,\ell)$. The control related term $f(t)$ is given by

$$f(t,x) = K_B \cdot (H''(x-x_1) - H''(x-x_2)) \cdot u(t), \qquad 0 \le x, x_1, x_2 \le \ell \tag{16}$$

where again $H(x)$ is the Heaviside function and $f(t)$ belongs to the Sobolev space (see [19]) $V^* = H^{-2}(0,\ell)$. If both $EI > 0$ and $c_D I > 0$, then σ_1 and σ_2 are V-elliptic and continuous with σ_1 symmetric; hence by Theorem 2 our beam equation is well posed for $f \in L_2((0,T),V^*)$ which includes the case of f given by (16).

3. PARAMETER ESTIMATION AND APPROXIMATION

As has been pointed out in the introduction, values of the coefficients of the damping and the piezoceramic coefficient control parameters (and the moduli of elasticity or rigidity for certain materials) are not well established. In the interest of obtaining accurate dynamic models and adequate control designs, these parameters must be estimated. The resulting parameter estimation problems for the beam with piezoceramic actuators and sensors can be stated in terms of finding parameters which give the best fit of the parameter dependent solutions of the partial differential equations to the observation data from response of the system to various excitations.

We assume then that the sesquilinear forms σ_1, σ_2 and the input or control terms f in section 2 are dependent upon parameters q with values in some admissible parameter metric space Q. Thus $\sigma_1 = \sigma_1(q), \sigma_2 = \sigma_2(q)$ and $f(t) = f(t;q)$ so that the corresponding solutions $y = y(q)$ of (6) or (10) are parameter dependent. We can thus consider the least squares estimation problem of minimizing over $q \in Q$ the least squares functional

$$J(q) = \sum_i |\, y(t_i;q) - z_i|^2 , \tag{17}$$

where $\{z_i\}$ are given observations and $\{y(t_i;q)\}$ are the parameter dependent mild solutions of (6) (or (10) or (11)) evaluated at each time t_i, $i = 1, 2, \cdots, \bar{N}$.

The minimization in our parameter estimation problems involves an infinite dimensional state (and an infinite dimensional admissible parameter space in some cases – for ease of exposition, we assume here that Q is finite dimensional). Motivated by computational requirements, we thus consider Galerkin type approximations in the context of the variational formulation of section 2. Let H^N be a sequence of finite dimensional subspaces of H (e.g, the span of finite sets of spline elements, or spectral elements, or modes, etc) that also satisfy $H^N \subset V$. We define orthogonal projections $P_H^N : H \to H^N$, the finite dimensional spaces $\mathcal{H}^N = H^N \times H^N$, and denote the orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}^N$ and $\mathcal{V}^*$ onto $\mathcal{H}^N$ by $P_{\mathcal{H}}^N$ and $P_{\mathcal{V}^*}^N$, respectively. Then the approximating estimation problems with finite dimensional state spaces can be stated as seeking $q \in Q$ which minimizes

$$J^N(q) = \sum_i \left| y^N(t_i;q) - z_i \right|^2 , \tag{18}$$

where $y^N(t;q)$ is the first coordinate of $w^N(t;q)$ given by

$$w^N(t;q) = \mathcal{T}^N(t;q)\, P_{\mathcal{H}}^N w_0(q) + \int_0^t \mathcal{T}^N(t-s;q)\, P_{\mathcal{V}^*}^N F(s)\, ds. \tag{19}$$

Here $\mathcal{T}^N(t;q)$ is the analytic semigroup generated by $\mathcal{A}^N(q)$ where $\mathcal{A}^N(q)$ is defined in the usual manner [1] via restriction of the (now parameter dependent) operators $\mathcal{A}(q)$ of (12) to $\mathcal{H}^N$, i.e., $\mathcal{A}^N(q) = P_{\mathcal{V}^*}^N \mathcal{A}(q) P_{\mathcal{H}}^N$.

We are now able to state an important result related to the convergence and continuous dependence (with respect to data) of the approximate solutions.

Theorem 3 *Let Q be a compact subset of $\mathcal{R}^N$ with metric d. Suppose both $\sigma_1(q)$ and $\sigma_2(q)$ are V-elliptic and continuous. Furthermore, assume $\sigma_1(q)$ and $\sigma_2(q)$ satisfy the continuity with respect to parameter condition*

$$| \sigma_i(q)(\phi, \psi) - \sigma_i(\tilde{q})(\phi, \psi) | \le \gamma_i \, d(q, \tilde{q}) \, |\phi|_V \, |\psi|_V, \qquad \text{for } \phi, \psi \in V \tag{20}$$

for $q, \tilde{q} \in Q$ and $i = 1, 2$. Furthermore assume that

$$q \to f(t; q) \text{ is continuous from } Q \text{ to } L_2((0,T), V^*) \tag{21}$$

and that H^N approximates V in the sense:

(H1) *For each $\phi \in V$, there exist $\phi^N \in H^N$ such that*

$$|\phi - \phi^N|_V \le \epsilon(N) \to 0 \qquad \text{as } N \to \infty.$$

Let q^N be arbitrary such that $q^N \to q$ in Q; then for $t > 0$

$$w^N(t; q^N) \to w(t; q) \qquad \text{in } \mathcal{V} \text{ norm.}$$

For a proof, see [3]. With Theorem 3, we obtain both parameter convergence and continuous dependence of parameter results (see [6] for discussions of parameter convergence and method stability as guaranteed by the convergence statement of this theorem). That is, the sequence of solutions $\bar{q}^N$ of the finite dimensional problems will converge (actually, in the general case, a subsequence converges) to a solution $\bar{q}$ of the original infinite dimensional problem, and for any observations $z^k \to z^0$, $dist(\bar{q}^N(z^k), \bar{q}(z^0)) \to 0$, as $k, N \to \infty$, where '$dist$' is the distance function between sets and $\bar{q}^N(z^k), \bar{q}(z^0)$ denote the sets of solutions of the least squares minimization problems corresponding to data z^k, z^0, respectively.

We note that the sesquilinear forms defined by (14) and (15) satisfy (20) if $q = (EI, c_D I, K_B)$ while $f(q)$ defined by (16) satisfies the required continuity condition (21). If we choose cubic splines for the basis of the approximation scheme, then (H1) is satisfied (again see [6]) and the desired convergence of Theorem 3 will be achieved.

4. CONTROL PROBLEMS

In this section we consider a linear quadratic regulator (LQR) formulation for the abstract system (11) of section 2. To this end, let Hilbert spaces U and Z be the input space and the output space, respectively. We consider the optimal control problem (R): Minimize the cost functional

$$J(w_0, u) = \int_0^\infty \left\{ | \mathcal{C} w(t)|_Z^2 + < Qu(t), u(t) >_U \right\} dt \tag{22}$$

over $u \in L_2((0, \infty), U)$ subject to

$$\begin{aligned} \dot{w}(t) &= \mathcal{A}w(t) + \mathcal{B}u(t) \qquad \text{in } \mathcal{V}^* \\ w(0) &= w_0. \end{aligned} \tag{23}$$

In (22), we assume that the observation operator $\mathcal{C} \in \mathcal{L}(\mathcal{H}, Z)$ (in this paper, we only consider the case when $\mathcal{C}$ is bounded, referring the reader to [5] for cases of unbounded $\mathcal{C}$), $Q \in \mathcal{L}(U,U) = \mathcal{L}(U)$ is self-adjoint and positive, and u is a U-valued control function. System (23) is the version of (11) with $F(t)$ replaced by $\mathcal{B}u(t)$ where $\mathcal{B} = (0, B)^T$ and B is an unbounded operator such that $B \in \mathcal{L}(U, V^*)$. The operator $\mathcal{A}$ is again given by (12)

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A_1 & -A_2 \end{pmatrix}$$

where A_1 and A_2 are defined through the sesquilinear forms σ_1 and σ_2 in (9). As above σ_1, σ_2 are V-elliptic and continuous. Then under standard assumptions (see [1]) this optimal control problem is well posed, i.e, given initial data $w_0 \in \mathcal{H}$ and $u \in L_2((0,\infty), U)$, the cost $J(w_0, u)$ is finite. In the following, we shall consider (23) in the mild solution form (see (13))

$$w(t) = \mathcal{T}(t)\, w_0 + \int_0^t \mathcal{T}(t-s)\, \mathcal{B}u(s)\, ds\,. \tag{24}$$

For the control problem (R), if $(\mathcal{A}, \mathcal{B})$ is stabilizable and $(\mathcal{A}, \mathcal{C})$ is detectable, then there exists a unique optimal solution $\bar{u}(t)$ for minimizing (22) given by

$$\bar{u}(t) = -Q^{-1}\mathcal{B}^*\Pi S(t) w_0, \tag{25}$$

where $\Pi \in \mathcal{L}(V^*, V)$ is the unique nonnegative solution to the operator algebraic Riccati equation

$$\mathcal{A}^*\Pi + \Pi\mathcal{A} - \Pi\mathcal{B}Q^{-1}\mathcal{B}^*\Pi + \mathcal{C}^*\mathcal{C} = 0 \tag{26}$$

and $S(t)$ is the exponentially stable semigroup generated by $\mathcal{A} - \mathcal{B}Q^{-1}\mathcal{B}^*\Pi$ (for details, see [5]).

Again motivated by computational considerations, we turn to approximation of the LQR problem (R). As in section 3, we approximate via Galerkin procedures. Let the sequence of finite dimensional subspaces $H^N \subset V \subset H$ and the associated orthogonal projections $P_H^N : H \to H^N$ be given as in section 3 and $\mathcal{H}^N = H^N \times H^N$, $P_{\mathcal{H}}^N : \mathcal{H} \to \mathcal{H}^N$. Define $\mathcal{A}^N : \mathcal{H}^N \to \mathcal{H}^N$ by the restriction of $\mathcal{A}$ to $\mathcal{H}^N$ and $\mathcal{C}^N$ by the restriction of $\mathcal{C}$ to $\mathcal{H}^N$. For a given $B \in \mathcal{L}(U, V^*)$, we define $B^N \in \mathcal{L}(U, H^N)$ by

$$< B^N u, \psi > = < u, B^*\psi > \qquad \text{for } \psi \in H^N,$$

and take $\mathcal{B}^N = (0, B^N)^T$. Then, the associated sequence of approximate LQR problems (R^N) are given by: Minimize

$$J^N(w_0^N, u) = \int_0^\infty \left\{ \left| \mathcal{C}^N w^N(t) \right|_Z^2 + < Qu(t), u(t) >_U \right\} dt \tag{27}$$

over $u \in L_2((0,\infty), U)$ subject to

$$w^N(t) = e^{t\mathcal{A}^N} w_0^N + \int_0^t e^{(t-s)\mathcal{A}^N}\, \mathcal{B}^N u(s)\, ds, \tag{28}$$

where $w_0^N = P_{\mathcal{H}}^N w_0$. If $(\mathcal{A}, \mathcal{B})$ is stabilizable and $(\mathcal{A}, \mathcal{C})$ is detectable, there exists a unique nonnegative self-adjoint solution Π^N to the algebraic Riccati equation in $\mathcal{H}^N$

$$\mathcal{A}^{N*}\Pi^N + \Pi^N\mathcal{A}^N - \Pi^N\mathcal{B}^N Q^{-1}\mathcal{B}^{N*}\Pi^N + \mathcal{C}^{N*}\mathcal{C}^N = 0. \tag{29}$$

The optimal control $\bar{u}^N$ for (R^N) is given by

$$\bar{u}^N(t) = -Q^{-1}\mathcal{B}^{N*}\Pi^N S^N(t) w_0^N \tag{30}$$

where $S^N(t) = e^{(\mathcal{A}^N - \mathcal{B}^N Q^{-1}\mathcal{B}^{N*}\Pi^N)t}$ on $\mathcal{H}^N$.

Before stating the main convergence theorem, we need to make one more assumption:

(H2) $V \hookrightarrow H$ is compact.

Theorem 4 *Assume that σ_1 and σ_2 are symmetric sesquilinear forms satisfying* (7) *and* (8). *Suppose $(\mathcal{A}, \mathcal{B})$ is stabilizable and $(\mathcal{A}, \mathcal{C})$ is detectable. Then as $N \to \infty$ we have*

$$\begin{aligned} \Pi^N P^N \xi &\to \Pi\xi && \text{in } \mathcal{V} \text{ for } \xi \in \mathcal{V}^* \\ S^N(t) P^N \xi &\to S(t)\xi && \text{in } \mathcal{V} \text{ for } \xi \in \mathcal{V}^*, \text{ and } t > 0, \end{aligned}$$

and hence $\bar{u}^N(t) \to \bar{u}(t)$.

For more details concerning these results, see [5]. We just note here that if we take the beam with piezoceramic actuator of section 1 and use cubic splines for the approximation elements, all of the conditions for the above results are met.

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REFERENCES

[1] H.T. Banks and J.A. Burns, *Introduction to Control of Distributed Parameter Systems*, Birkhaüser, Boston, 1992.

[2] H.T. Banks and K. Ito, Approximation in LQR problems for infinite dimensional systems with unbounded input operators, *Proc. Conf. on Optimization*, Haifa, 1992, *to appear.*

[3] H.T. Banks, K. Ito and Y. Wang, Well-posedness and approximation for damped second order systems with unbounded input operators, *to appear.*

[4] H.T. Banks and K. Kunisch, *Estimation Techniques for Distributed Parameter Systems*, Birkhaüser, Boston, 1989.

[5] E.F. Crawley and E.H. Anderson, Detailed models for piezoceramic actuation of beams, AIAA Conf. paper 89-1388-CP, 1989, pp. 2000-2010.

[6] R.L. Clark, Jr., C.R. Fuller and A Wicks, Characterization of multiple piezoelectric actuators for structural excitation, *J. Acoust. Soc. Amer.*, 1991, *to appear.*

[7] E.K. Dimitriadis, C.R. Fuller and C.A. Rogers, Piezoelectric actuators for distributed noise and vibration excitation of thin plates, *Proc. 8th ASME Conf. on Failure Prevention, Reliability, and Stress Analysis*, Montreal, 1989, pp.223-233.

[8] J.L. Fanson and T.K. Caughey, Positive position feedback control for large structures, *Structural Dynamics and Materials Conf.*, AIAA Conf. Paper 87-0902, 1987, pp. 588-598.

[9] R.E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Pitman, London, 1977.

[10] H. Tanabe, *Equations of Evolution*, Pitman, London, 1979.

[11] J. Wloka, *Partial Differential Equations*, Cambridge University Press, New York, 1987.